Incompatibility between Carnot efficiency and finite power in Markovian dynamics

Naoto Shiraishi[1] and Keiji Saito[2]

[1] *Department of Basic Science, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8902, Japan*

[2] *Department of Physics, Keio University, 3-14-1 Hiyoshi, Yokohama 223-8522, Japan*

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In Markovian dynamics with the local detailed balance condition, we decompose the total entropy production rate into microscopic transitions. By applying this decomposition to the heat to work conversion process, we rigorously show that the Carnot efficiency implies zero power for any heat engine, even with broken time-reversal symmetry beyond the linear response regime. Moreover, we propose a trade-off relationship between the entropy production rate and the heat flow between the system and bath.

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Introduction.— Heat to work conversion has been a central subject in thermodynamics. Typical mechanisms for extracting work from heat are the cyclic heat engine [1] and thermoelectric transport [2] (see Fig. 1). These are categorized as different mechanisms in the sense that the former includes periodically changing control parameters such as external forces and environmental temperature changes, while in the latter, no time-dependent parameters are included and the work is provided as a steady state electric current through thermoelectric materials.

In recent years considerable effort has been devoted to finding thermoelectric materials with higher efficiency [3–6]. Stochastic cyclic heat engines in small systems have attracted attention because such small heat engines can be scrutinized through precise measurements [7–11]. Therefore, it is a critical and urgent problem to understand the mechanisms underlying the heat to work conversion processes in the light of recent progress in nonequilibrium statistical mechanics [12].

The thermodynamic efficiency in one cycle between inverse temperatures β_H and β_C ($\beta_H < \beta_C$) is bounded by the celebrated Carnot efficiency [13, 14]

$$\eta_C \ = \ 1 - \frac{\beta_H}{\beta_C} \,. \tag{1}$$

So far, there is considerable research on the fundamental relation between the efficiency and power [15–35]. Benenti and coworkers developed an argument on thermodynamic efficiency using a quite general analysis within the linear response regime for thermoelectric transport and showed that broken time-reversal symmetry (i.e., nonsymmetrical Onsager matrix) could in principle increase the thermodynamic efficiency, and even devices operating reversibly at finite power seem to be realizable [20]. At this level of argument, the restriction on the Onsager matrix elements imposed by the second law does not prohibit the coexistence of finite power with the reversibility condition (i.e., zero total entropy production). This has triggered a number of studies based on specific dynamical models to investigate the relation between the power and efficiency for systems with broken-time reversal symmetry [21–33]. Multi-terminal thermoelectric transport in the presence of a magnetic field was studied and a more stringent bound was discovered, which prohibits the coexistence of finite power and the Carnot efficiency [21]. More recently, general frameworks of the linear response theory have been developed for stochastic heat engines where the system is periodically driven by changing the control parameters, and the dynamics of the distribution obeys the Fokker-Planck equation [31–33]. Onsager matrices are also defined for such periodically driven systems [16, 17], which are generally nonsymmetrical, similar to the case of thermoelectric transport with a magnetic field. A detailed analysis has revealed the exact bound between power $\dot{W}$ and efficiency η as $\dot{W} \leq N(\eta/\eta_C)(1-(\eta/\eta_C))$, where N is a constant [31]. This clearly shows zero power at the Carnot efficiency. Recently a similar argument has also been developed with isothermal heat engines [32].

From these studies in the linear response regime, one anticipates the general mechanism for incompatibility between finite power and the Carnot efficiency regardless of the broken time-reversal symmetry even beyond the linear response regime. The aim of this paper is to understand this mechanism with Markovian dynamics for any system, even ones with broken time-reversal symmetry. To this end, we used the idea of decomposition of the total entropy production into *partial entropy production*, which was first introduced to consider the microscopic thermodynamic structure in jump processes with feedback controls [36–38]. The partial entropy production connects the information for total entropy production to the detailed properties of individual transitions and the probability distribution. By generalizing this and applying it to heat to work conversion, we prove the no-go theorem that systems at the Carnot efficiency possess no power for stochastic heat engines and thermoelectric transport. In addition, we demonstrate that the amount of entropy production rate gives an upper bound for heat

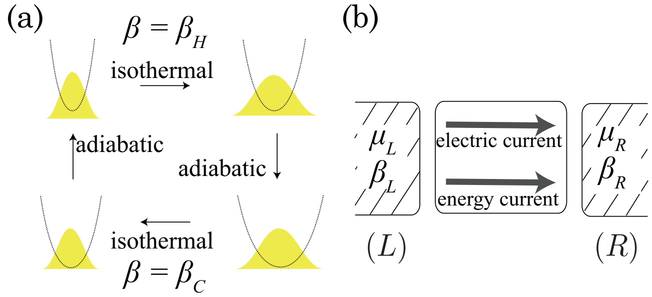

FIG. 1: (color online) Schematic picture of typical heat to work converting mechanisms, (a): stochastic heat engine (b): thermoelectric transport. In (a), an example of a stochastic heat engine is shown, which is discussed in Ref. [19]. A Brownian particle is trapped by a time-dependent harmonic potential, and yellow parts are the distribution of the particle. In (b), β_α and μ_α ($\alpha = L, R$) are respectively the inverse temperatures and chemical potentials of the α-th electrode satisfying $\beta_L^{-1} > \beta_R^{-1}$ and $\mu_L < \mu_R$. Energy current drives finite electric current.

flow between the system and a reservoir in the linear response regime.

Stochastic heat engine.— We consider the stochastic heat engine where classical Brownian particles are controlled in one cycle. In general, the thermodynamic process is divided into isothermal processes at fixed temperatures and adiabatic processes. In isothermal processes, one controls the system's time parameter λ_t. A well-known theoretical model is the one introduced by Schmiedl and Seifert [19], which is schematically shown in Fig. 1(a). In the Schmiedl-Seifert engine, a particle is trapped by a harmonic potential, and the shape of the trap potential is changed in time. Having such an example in mind, we make a general argument on heat dissipation and power generation.

Since adiabatic processes cause no entropy production, the averaged total entropy production during a single cycle $\Delta S_{\rm tot}$ is given by summing all contributions from isothermal processes

$$\Delta S_{\rm tot} \;=\; \sum_i \Delta S_{\rm tot}^{(i)} \,, \tag{2}$$

where $\Delta S_{\rm tot}^{(i)}$ represents the amount of entropy production during the i-th isothermal process of temperature β_i^{-1}. The power $\dot W$ for one cycle is given by the relation

$$\dot W \;=\; -\frac{1}{\tau}\int_0^\tau dt \dot Q \,, \tag{3}$$

where τ represents the period of one cycle, and $\dot Q$ represents the heat flow from the engine to the reservoir. Here, we use the fact that the initial and final distributions are identical in the steady state cycle. We assume that the stochastic dynamics are Markovian. Time-evolution of the distribution in the i-th isothermal process is given by the master equation [39]:

$$\frac{\partial}{\partial t} P_t(w) \;=\; \int dw' L^{(i)}_{w\,w'}(\lambda_t) P_t(w') \,, \tag{4}$$

where w represents the positions and velocities of the particles and λ_t stands for the control parameter. $P_t(w)$ represents a probability of the state w at time t.

Typical Markovian dynamics, such as the Fokker-Planck equation, possess the local detailed balance condition [39]

$$\frac{L^{(i)}_{ww'}(\lambda_t)}{\tilde L^{(i)}_{\tilde w' \tilde w}(\lambda_t)} \;=\; e^{-\beta_i(\epsilon_{\lambda_t}(w)-\epsilon_{\lambda_t}(w'))} \,, \tag{5}$$

where $\epsilon_{\lambda_t}(w)$, $\tilde w$, and $\tilde L$ represent the energy at the state w with λ_t, the time-reversal of the state w, and the generator with an inverse magnetic field, respectively. The local detailed balance condition reflects the time-symmetric property of equilibrium states that, in the equilibrium condition, a stochastic trajectory and its time-reversal trajectory with a reversed magnetic field occur with the same probability. The total entropy production rate is given by $\dot S^{(i)}_{\rm tot} = \int dw \dot P_t(w)[-\log P_t(w) - \epsilon_{\lambda_t}(w)]$, and using the local detailed balance condition it is also expressed in the following form [12, 40]

$$\dot S^{(i)}_{\rm tot} \;=\; \int dw dw' L^{(i)}_{w\,w'}(\lambda_t) P_t(w') \log\left[\frac{L^{(i)}_{w\,w'}(\lambda_t) P_t(w')}{\tilde L^{(i)}_{\tilde w'\,\tilde w}(\lambda_t) P_t(w)}\right] , \tag{6}$$

where the integration over w and w' are performed for all possible transition processes with $L^{(i)}_{w\,w'}(\lambda_t) \neq 0$. It is straightforward to check the nonnegativity for this expression.

Partial entropy production and zero power.— We here consider decomposition of the total entropy production rate into microscopic transition processes [36]. We introduce a general expression of the *partial entropy production rate* for systems with broken time-reversal symmetry, which assigns the entropy production rate to the process from w' to w as

$$\begin{aligned}\dot S^{(i)}_{w\,w'} \;=\; & L^{(i)}_{w\,w'}(\lambda_t) P_t(w') \log\left[\frac{L^{(i)}_{w\,w'}(\lambda_t) P_t(w')}{\tilde L^{(i)}_{\tilde w'\,\tilde w}(\lambda_t) P_t(w)}\right] \\ & + \tilde L^{(i)}_{\tilde w'\,\tilde w}(\lambda_t) P_t(w) - L^{(i)}_{w\,w'}(\lambda_t) P_t(w') \,. \end{aligned} \tag{7}$$

This quantity satisfies two key ingredients that ensure qualification of reasonable assignment of the entropy production rate to microscopic processes. The first property is that summing over all transition processes reproduces the total entropy production rate:

$$\dot S^{(i)}_{\rm tot} \;=\; \int dw dw' \, \dot S^{(i)}_{w\,w'} \,. \tag{8}$$

This is a consequence of the conservation of the probability as $\int dw' \tilde{L}^{(i)}_{\tilde{w}'\,\tilde{w}}(\lambda_t) = \int dw L^{(i)}_{w\,w'}(\lambda_t) = 0$. The second property is the non-negativity:

$$\dot{S}^{(i)}_{w\,w'} \;\geq\; 0\,, \tag{9}$$

from a simple mathematical relation $a\log(a/b)+b-a \geq 0$ for arbitrary positive a and b. The equality holds if and only if $a=b$. From these properties, the partial entropy production rate (7) gives a physically reasonable assignment of dissipation to a single transition $w' \to w$.

Now we discuss the universal mechanism to achieve zero power at the Carnot efficiency. The Carnot efficiency can be reached at zero total entropy production for one cycle, i.e., $\Delta S_{\rm tot} = 0$. From Eqs.(2), (8) and (9), zero total entropy production eventually implies $\dot{S}^{(i)}_{w\,w'} = 0$ for all transition processes at any time. We notice that the equality in (9) holds only when the detailed balance condition is satisfied:

$$\frac{P_t(w)}{P_t(w')} = \frac{L^{(i)}_{ww'}(\lambda_t)}{\tilde{L}^{(i)}_{\tilde{w}'\tilde{w}}(\lambda_t)} = \frac{e^{-\beta_i \epsilon_{\lambda_t}(w)}}{e^{-\beta_i \epsilon_{\lambda_t}(w')}}\,. \tag{10}$$

Note that this must be satisfied for any possible transition process. We now evaluate the heat flow $\dot{Q}$. From the master equation (4), $\dot{Q}$ in the i-th isothermal process is calculated as

$$\begin{aligned}\dot{Q} &= -\int dw \epsilon_{\lambda_t}(w) \int dw' L^{(i)}_{w\,w'}(\lambda_t) P_t(w') \\ &= -\int dw \epsilon_{\lambda_t}(w) P_t(w) \int dw' \tilde{L}^{(i)}_{\tilde{w}'\tilde{w}}(\lambda_t) \\ &= 0. \end{aligned} \tag{11}$$

In the second and third lines, we use the detailed balance condition (10) and the conservation of probability, respectively. $\dot{Q}=0$ in (3) implies zero power. Thus, the reversibility condition is connected to zero power even for systems with broken time-reversal symmetry. This is our main result. We emphasize that the derivation of zero power highly depends on the local detailed balance condition and the properties of the partial entropy production rate.

Thermoelectric transport.— Our analysis is valid not only for stochastic heat engines, but also for thermoelectric transport as long as the dynamics can be mapped onto the classical Markovian probabilistic process. We demonstrate this by considering steady state transport through a system with finite lattices in the presence of a magnetic field. Suppose that an electric conductor is attached to the left and right electrodes which have different temperatures and chemical potentials denoted by β_α^{-1} and μ_α respectively for the α-th electrode ($\alpha = L, R$). See Fig. 1(b) for a schematic picture. We assume the following standard setup: Electrodes are modeled by free electrons, and electrodes and the system are connected via the tight-binding coupling interaction. Then, we employ the dynamics given by the Pauli master equation [41], where the transition rate between eigenstates of the system is calculated using Fermi's golden rule [42]. Let $P_t(k)$ be the diagonal element of the density matrix in the representation of the k-th eigenstate of the system. The Pauli master equation is given by

$$\frac{\partial}{\partial t} P_t(k) \;=\; \sum_{\alpha=L,R} \sum_{\ell} L_{\alpha,k\,\ell} P_t(\ell)\,, \tag{12}$$

where $L_{\alpha,k\,\ell}$ stands for the transition rate from the state ℓ to k via the effect of the α-th electrode. Let $\tilde{L}$ be the transition rate matrix for the dynamics with a reversed magnetic field and let $\tilde{k}$ be the time-reversal of the k-th eigenstate. Then, from the argument for the time-reversal symmetry in the equilibrium state, the detailed balance condition is imposed as $L_{\alpha,k\,\ell} P_{{\rm eq},\alpha}(\ell) = \tilde{L}_{\alpha,\tilde{\ell}\,\tilde{k}} P_{{\rm eq},\alpha}(k)$, where $P_{{\rm eq},\alpha}(\ell)$ is the equilibrium distribution with β_α and μ_α; $P_{{\rm eq},\alpha}(\ell) = e^{-\beta_\alpha(\epsilon_\ell - \mu_\alpha n_\ell)}/Z$. From this the local detailed balance condition is given by

$$\frac{L_{\alpha,k\,\ell}}{\tilde{L}_{\alpha,\tilde{\ell}\,\tilde{k}}} \;=\; e^{-\beta_\alpha[(\epsilon_k - \epsilon_\ell) - \mu_\alpha(n_k - n_\ell)]}\,. \tag{13}$$

This can also be given by the detailed expression of the transition rate derived with a specific setup for the total Hamiltonian [42]. We should also note that the Pauli master equation is equivalent to the dynamics for the diagonal elements in the Lindblad quantum master equation that is derived via the standard procedure with the Born-Markov and secular approximations with a microscopic Hamiltonian [43].

From the continuity equation with respect to energy $\partial_t[\sum_k \epsilon_k P_t(k)] = \sum_\alpha \sum_{k,\ell} \epsilon_k L_{\alpha,k\,\ell} P_t(\ell)$, the energy current into the electrodes is expressed as $J_{\epsilon,\alpha} = -\sum_{k,\ell}(\epsilon_k - \epsilon_\ell) L_{\alpha,k\,\ell} P_t(\ell)$. Similarly, we obtain the expression of the electron current as $J_{\rho,\alpha} = -\sum_{k,\ell}(n_k - n_\ell) L_{\alpha,k\,\ell} P_t(\ell)$. At the steady state, the total entropy production rate is generated only from the heat flow into the reservoirs, and is given by $\dot{S}_{\rm tot} = \sum_\alpha \beta_\alpha (J_{\epsilon,\alpha} - \mu_\alpha J_{\rho,\alpha})$, where the currents are steady state currents. Using the local detailed balance condition (13), it can be written in the following form

$$\dot{S}_{\rm tot} \;=\; \sum_{\alpha=L,R} \sum_{k,\ell} L_{\alpha,k\,\ell} P_{ss}(\ell) \log\left[\frac{L_{\alpha,k\,\ell} P_{ss}(l)}{\tilde{L}_{\alpha,\tilde{\ell}\,\tilde{k}} P_{ss}(k)}\right]\,, \tag{14}$$

where P_{ss} is the steady state distribution. This form is an extension of the expression for the stochastic heat engine (6) to multiple reservoirs. In a similar manner to (7), we introduce the *partial entropy production rate*

$$\begin{aligned}\dot{S}_{k,\ell} \;=\; &\sum_{\alpha=L,R} L_{\alpha,k\,\ell} P_{ss}(\ell) \log\left[\frac{L_{\alpha,k\,\ell} P_{ss}(\ell)}{\tilde{L}_{\alpha,\tilde{\ell}\,\tilde{k}} P_{ss}(k)}\right] \\ &+ \tilde{L}_{\alpha,\tilde{\ell}\,\tilde{k}} P_{ss}(k) - L_{\alpha,k\,\ell} P_{ss}(\ell)\,, \end{aligned} \tag{15}$$

where we can easily check the nonnegativity and that the summation reproduces the total entropy production rate.

Now, we impose the zero entropy production condition. Following the same argument as in the stochastic heat engine, we eventually end up with the detailed balance conditions

$$\frac{P_{ss}(k)}{P_{ss}(\ell)} = \frac{e^{-\beta_\alpha(\epsilon_k-\mu_\alpha n_k)}}{e^{-\beta_\alpha(\epsilon_\ell-\mu_\alpha n_\ell)}}, \quad \alpha = L, R. \tag{16}$$

For the case $\alpha = L$, we have $P_{ss} = P_{\mathrm{eq},L}$ while the case $\alpha = R$ leads to $P_{ss} = P_{\mathrm{eq},R}$. These must be satisfied simultaneously. This leads to no energy and electron currents from each reservoir. Thus, we get zero power as a result of the reversibility condition irrespective of the amplitude of a magnetic field. Equation (16) can be satisfied for resonant tunneling where electrons are transmitted via only one energy window, such as single quantum-dot transport [44–47].

Bound for entropy production rate.— So far, we rigorously showed the fundamental principle that a zero total entropy production rate leads to zero heat flow into a thermal reservoir, even for systems with broken time-reversal symmetry. This simultaneously implies that finite heat flow inevitably causes finite total entropy production. Note that finite heat flow is necessary to get finite power. Then as a next step, we consider the impact of finite heat flow on the total entropy production rate. We show that finite heat flow gives a lower bound for the total entropy production rate in the linear response regime. In this part, we suppose that all possible transitions are associated with a single reservoir for simplicity.

We first introduce a quantity characterizing the deviation from the detailed balance condition defined as

$$\Delta_{w,w'} = 1 - \frac{L_{w\,w'}P(w')}{\tilde{L}_{\tilde{w}'\,\tilde{w}}P(w)}, \tag{17}$$

which is small due to the assumption of the linear regime, and satisfies the following constraint:

$$\int dwdw' \tilde{L}_{\tilde{w}'\tilde{w}}P(w)\Delta_{w,w'} = 0. \tag{18}$$

The partial entropy production rates are evaluated as $\dot{S}_{ww'} = \tilde{L}_{\tilde{w}'\tilde{w}}P(w)\Delta^2_{w,w'} + O(\Delta^3_{w,w'})$. Using this, the total entropy production rate is calculated up to $O(\Delta^2)$ as

$$\dot{S}_{\mathrm{tot}} = \int dwdw' \tilde{L}_{\tilde{w}'\tilde{w}}P(w)\Delta^2_{w,w'}. \tag{19}$$

The heat flow into the reservoir is calculated as

$$\begin{aligned}\dot{Q} &= -\int dwdw' \epsilon(w) L_{w\,w'}P(w') \\ &= \int dwdw' \epsilon(w)P(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta_{w,w'}.\end{aligned} \tag{20}$$

Now we derive the lower bound of the entropy production rate (19) under a constraint (18) and a fixed heat flow (20). We apply the method of Lagrange multipliers for $\Delta_{w,w'}$ with $w \neq w'$ by considering $\Delta_{w,w'}$ as if they are independent of each other for all (w, w') [48]. We here exclude the case of $w = w'$ because $\Delta_{w,w'}$ is always zero for $w = w'$. Since the difference between the probability distribution $P(w)$ and the equilibrium $P_{\mathrm{eq}}(w)$ is $O(\Delta)$, we can replace $P(w)$ with $P_{\mathrm{eq}}(w)$, and then arrive at the relation

$$\dot{S}_{\mathrm{tot}} \geq \frac{1}{C}\dot{Q}^2. \tag{21}$$

Here C is a constant

$$C = \int dw \epsilon(w)^2 A(w) - \frac{\left(\int dw \epsilon(w) A(w)\right)^2}{\int dw A(w)}. \tag{22}$$

with the dynamical activity in equilibrium [49]: $A(w) = \mathcal{P}\int_{w\neq w'} dw' P_{\mathrm{eq}}(w) L_{w'w}$, where $\mathcal{P}$ is the principal value integral. The inequality (21) clearly shows that the heat flow possesses at least a quadratic contribution to the entropy production rate. In other words, for a given entropy production rate the heat flow is bounded as $\sqrt{C\dot{S}_{\mathrm{tot}}} \geq |\dot{Q}|$. We remark that the bound is derived by neglecting correlations of $\Delta_{w,w'}$ between different (w, w'), and hence our bound $\dot{Q}^2/C$ is lower than a true lower bound. Except for dynamics with discrete states, the equality is in general difficult to achieve.

Summary.— In this Letter, we applied the idea of partial entropy production in Markovian heat to work converting systems. The partial entropy production rate is a decomposition of the total entropy production rate and satisfies the nonnegativity. These properties lead to the fact that heat to work conversion systems in general never attain the Carnot efficiency with finite power. This resolves controversies raised by the Onsager matrix argument for a system with broken time-reversal symmetry [20], as long as the dynamics are Markovian. We here stress that the local detailed balance is crucial in our argument. Our approach generalizes the recent exact studies in the linear response regime using the Fokker-Planck equation [31–33].

The power of heat engines has been investigated with linear irreversible thermodynamics and is expressed in terms of Onsager matrix coefficients [15, 20, 23]. It has recently been suggested that there may be a stringent bound for the Onsager matrix even if the off-diagonal coefficients are asymmetric due to a magnetic field [23]. It is an open and intriguing problem if the method of the partial entropy production can provide an insight to obtain a new bound on irreversible thermodynamics.

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[1] K. Sekimoto, *Stochastic Energetics*, (Springer 2010).
[2] G. Benenti, G. Casati, T. Prosen and K. Saito, arXiv:1311.4430v1 (2013).
[3] G. Mahan, B. Sales, and J. Sharp, Phys. Today **50**, 42 (1997).
[4] A. Majumdar, Science **303**, 777 (2004).
[5] M.S. Dresselhaus, G. Chen, M.Y. Tang, R.G. Yang, H. Lee, D.Z. Wang, Z.F. Ren, J.-P. Fleurial, and P. Gogna, Adv. Mater. **19**, 1043 (2007).
[6] G.J. Snyder and E.R. Toberer, Nature Materials **7**, 105 (2008).
[7] P. G. Steeneken, K. Le Phan, M. J. Goossens, G. E. J. Koops, G. J. A. M. Brom, C. van der Avoort and J. T. M. van Beek, Nat. Phys. **7**, 354 (2010).
[8] V. Blickle and C. Bechinger, Nat. Phys. **8**, 143 (2011).
[9] M. Ribezzi-Crivellari and F. Ritort, Proc. Natl. Acad. Sci. USA **111**, E3386 (2014).
[10] J .V. Koski, V. F. Maisi, J. P. Pekola and D. V. Averin, Proc. Natl. Acad. Sci. USA **111**, 13786 (2014).
[11] J. Rosnagel, S.T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. S. Kaler and Kilian Singer, arXiv:1510.03681 (2015).
[12] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[13] S. Carnot, *Reflections on the Motive Power of Fire and on Machines Fitted to Develop that Power*, Paris: Bachelier (1824).
[14] H. B. Callen, *Thermodynamics and an Introduction to Thermostatics*, 2nd ed. (John Wiley & Sons, New York, 1985).
[15] C. Van den Broeck, Adv. Chem. Phys. **135**, 189 (2007).
[16] Y. Izumida and K. Okuda, Phys. Rev. E **80**, 021121 (2009).
[17] Y. Izumida and K. Okuda, Euro. Phys. J. B **77**, 499 (2010).
[18] M. Esposito, R. Kawai, K. Lindenberg, and C. Van den Broeck, Phys. Rev. Lett. **105**, 150603 (2010).
[19] T. Schmiedl and U. Seifert, Europhys. Lett. **81**, 20003 (2011).
[20] G. Benenti, K. Saito, and G. Casati, Phys. Rev. Lett. **106**, 230602 (2011).
[21] K. Brandner and U. Seifert, New J. Phys. **15**, 105003 (2013).
[22] B. Sothmann and M. Büttiker, Europhys. Lett. **99**, 27001 (2012).
[23] K. Brandner, K. Saito, and U. Seifert, Phys. Rev. Lett. **110**, 070603 (2013).
[24] V. Balachandran, G. Benenti, and G. Casati, Phys. Rev. B **87**, 165419 (2013).
[25] A. E. Allahverdyan, K. V. Hovhannisyan, A. V. Melkikh, and S. G. Gevorkian, Phys. Rev. Lett. **111**, 050601 (2013).
[26] J. Stark, K. Brandner, K. Saito, and U. Seifert, Phys. Rev. Lett. **112**, 140601 (2014).
[27] K. Brandner and U. Seifert, Phys. Rev. E **91**, 012121 (2015).
[28] R. Sánchez, B. Sothmann, and A. N. Jordan, Phys. Rev. Lett. **114**, 146801 (2015).
[29] M. Mintchev, L. Santoni, and P. Sorba, arXiv:1310.2392 (2013).
[30] V. Holubec, J. Stat. Mech. P05022 (2014).
[31] K. Brandner, K. Saito, and U. Seifert, Phys. Rev. X **5**, 031019 (2015).
[32] K. Proesmans and C. Van den Broeck, Phys. Rev. Lett. **115**, 090601 (2015).
[33] K. Proesmans, B. Cleuren, and C. Van den Broeck, arXiv:1511.03135 (2015).
[34] K. Sekimoto and S.-i. Sasa, J. Phys. Soc. Jpn. **66**, 3326 (1997).
[35] E. Aurell, K. Gawędzki , C. Mejía-Monasterio, R. Mohayaee, P. Muratore-Ginanneschi, J. Stat. Phys. **147**, 487 (2012).
[36] N. Shiraishi and T. Sagawa, Phys. Rev. E **91**, 012130 (2015).
[37] N. Shiraishi, S. Ito, K. Kawaguchi, and T. Sagawa, New J. Phys. **17**, 045012 (2015).
[38] N. Shiraishi, T. Matsumoto, and T. Sagawa, New J. Phys. **18**, 013044 (2016).
[39] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer, 1989).
[40] A. Gomez-Marin, J. M. R. Parrondo, and C. Van den Broeck, Phys. Rev. E **78**, 011107 (2008).
[41] H. J. Kreuzer, *Nonequilibrium thermodynamics and its statistical foundations*, (Oxford University Press 1981).
[42] Suppose that the total Hamiltonian is given by $H_T = H + \sum_{\alpha=L,R} H_\alpha + H_{\alpha S}$, where H is a system's Hamiltonian and H_α is the Hamiltonian of the α-th electrode. $H_{\alpha S}$ stands for the coupling between the α-th electrode and the system. The electrodes are an ensemble of free electrons $H_\alpha = \sum_{k\,\sigma} \epsilon_k c^\dagger_{\alpha,k\,\sigma} c_{\alpha,k\,\sigma}$, where $c_{\alpha,k\,\sigma}$ is the annihilation fermion operator of the wave number vector k and spin σ. As in the standard setup, we assume that the free electrons in the electrodes couple with the system via the tight-binding coupling Hamiltonian, e.g., $H_{\alpha S} = \sum_s \sum_{k,\sigma} \gamma_{\alpha,k\,s} c^\dagger_{\alpha,k\,\sigma} d_{s\sigma} + \gamma^*_{\alpha,k\,s} d^\dagger_{s\sigma} c_{\alpha,k\,\sigma}$, where $d_{s\sigma}$ is the annihilation fermion operator at the site s and spin σ in the system. Let $|k\rangle$ be the k-th eigenstate of the system and let ρ_α be the equilibrium density matrix in the α-th electrode. Then the transition rate $L_{\alpha,k\,\ell}$ is given by Fermi's golden rule
$$L_{\alpha,k\,\ell} = \frac{2\pi}{\hbar} \mathrm{Tr}\left[\mathcal{P}_\ell H_{\alpha S} \rho_\alpha \mathcal{P}_k H_{\alpha S} \mathcal{P}_\ell\right] ,$$
where $\mathcal{P}_k = |k\rangle\langle k|$.
[43] H. Breuer and F. Petruccione, *Theory of Open Quantum Systems*, (Oxford, Oxford, 2002).
[44] G.D. Mahan, J.O. Sofo, Proc. Natl. Acad. Sci. USA **93**, 7436. (1996).
[45] T.E. Humphrey, R. Newbury, R.P. Taylor, H. Linke, Phys. Rev. Lett. **89**, 116801 (2002).
[46] T.E. Humphrey, H. Linke, Phys. Rev. Lett. **94**, 096601 (2005).
[47] N. Shiraishi, Phys. Rev. E **92**, 050101 (2015).
[48] See Supplemental Material.
[49] C. Maes, K. Netočný, and B. Wynants, J. Phys. A: Math. Theor. **45**, 455001 (2012).

Supplemental Material

Derivation of inequality (21)

We show the derivation of the inequality (21) by using the method of Lagrange multipliers. Our original problem is to derive a lower bound for Eq. (19) by tuning the probability distribution $P(w)$ under a constraint Eq. (20) with given $\dot{Q}$. Note that $\Delta_{w,w'}=0$ with $w=w'$ for any probability distribution $P(w)$, and thus $\Delta_{w,w'}$ at $w=w'$ has no contribution in Eqs. (18), (19), and (20). Here, we solve the following optimizing problem: We derive a bound for Eq. (19) by tuning $\Delta_{w,w'}$ except for $w=w'$ without taking account of correlations of $\Delta_{w,w'}$ between different (w,w'). Because we neglect the correlations, our lower bound for $\dot{S}_{\rm tot}$ is weaker than a true bound. We however emphasize that the derived bound is still a bound for $\dot{S}_{\rm tot}$ and is useful to understanding the relation between the total entropy production rate and heat flow.

We minimize the total entropy production rate

$$\dot{S}_{\rm tot}=\mathcal{P}\int_{w\neq w'}dwdw'P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta_{w,w'}^2 \tag{S.1}$$

under the constraint

$$0=\mathcal{P}\int_{w\neq w'}dwdw'P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta_{w,w'}, \tag{S.2}$$

$$\dot{Q}=\mathcal{P}\int_{w\neq w'}dwdw'\epsilon(w)P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta_{w,w'} \tag{S.3}$$

with fixed $\dot{Q}$. Here, the representation of the integration $\mathcal{P}\int_{w\neq w'}dwdw'$ is used in the sense of the principal value integral.

By introducing the Lagrange multipliers λ and η, our problem is reduced to a problem of minimizing the following quantity

$$\begin{aligned}F(\Delta_{w,w'},\lambda,\eta)=&\mathcal{P}\int_{w\neq w'}dwdw'P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta_{w,w'}^2+\lambda\mathcal{P}\int_{w\neq w'}dwdw'P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta_{w,w'}\\ &+\eta\mathcal{P}\int_{w\neq w'}dwdw'\epsilon(w)P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta_{w,w'}-\eta\dot{Q}.\end{aligned} \tag{S.4}$$

First the differentiation of F by $\Delta_{w,w'}$ equals zero with the argument of the minimum $\Delta^*_{w,w'}$, λ^*, and η^*:

$$2P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}\Delta^*_{w,w'}+\lambda^*P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}+\eta^*\epsilon(w)P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}=0. \tag{S.5}$$

Then by operating $\mathcal{P}\int_{w\neq w'}dwdw'$ to Eq. (S.5), we obtain

$$\lambda^*=-\eta^*\frac{\mathcal{P}\int_{w\neq w'}dwdw'\epsilon(w)P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}}{\mathcal{P}\int_{w\neq w'}dwdw'P_{\rm eq}(w)\tilde{L}_{\tilde{w}'\tilde{w}}}=-\eta^*\frac{\int dw\epsilon(w)A(w)}{\int dwA(w)}, \tag{S.6}$$

where we used $\mathcal{P}\int_{w\neq w'}dw'\tilde{L}_{\tilde{w}'\tilde{w}}=\mathcal{P}\int_{w\neq w'}dw'L_{w'w}$. By substituting Eq. (S.6) into Eqs. (S.3) and (S.5), both η^* and $\Delta^*_{w,w'}$ are derived as

$$\Delta^*_{w,w'}=\frac{\eta^*}{2}\left(\frac{\int dw\epsilon(w)A(w)}{\int dwA(w)}-\epsilon(w)\right), \tag{S.7}$$

$$\eta^*=\frac{2\dot{Q}}{\dfrac{\left(\int dw\epsilon(w)A(w)\right)^2}{\int dwA(w)}-\int dw\epsilon(w)^2A(w)}, \tag{S.8}$$

which leads to the lower bound (21).